

Extensions to Network Flow Interdiction on Planar Graphs

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Abstract

Network flow interdiction analysis studies by how much the value of a maximum flow in a network can be diminished by removing components of the network constrained to some budget. Although this problem is strongly NP-complete on general networks, pseudo-polynomial algorithms were found for planar networks with a single source and a single sink and without the possibility to remove vertices. In this work we introduce pseudo-polynomial algorithms which overcome some of the restrictions of previous methods. We propose a planarity-preserving transformation that allows to incorporate vertex removals and vertex capacities in pseudo-polynomial interdiction algorithms for planar graphs. Additionally, a pseudo-polynomial algorithm is introduced for the problem of determining the minimal interdiction budget which is at least needed to make it impossible to satisfy the demand of all sink nodes, on planar networks with multiple sources and sinks satisfying that the sum of the supplies at the source nodes equals the sum of the demands at the sink nodes. Furthermore we show that the k -densest subgraph problem on planar graphs can be reduced to a network flow interdiction problem on a planar graph with multiple sources and sinks and polynomially bounded input numbers. However it is still not known if either of these problems can be solved in polynomial time.

1 Introduction

In this paper we are interested in minimizing the maximum flow of a network by removing arcs and vertices constrained to some interdiction budget. This problem is known as *network interdiction* or *network flow interdiction*¹. One can either allow or disallow partial removal of arcs (removing half of an edge corresponds to reduce its capacity to half of the original value). However the techniques and results seem not to substantially differ on this issue. We are interested in the case without partial arc destruction.

The problem of finding the k *most vital arcs* of a flow network is a special case of the network interdiction problem where k arcs have to be removed such that the maximum flow is reduced as much as possible. Network interdiction and related problems appear in various areas such as drug interdiction [19], military planning [8], protecting electric power grids against terrorist attacks [17] and hospital infection control [2].

The network interdiction problem was shown to be strongly NP-complete on general graphs and weakly NP-complete when restricted to planar graphs [15, 19]. Different algorithms for finding exact solutions were proposed ([8, 13, 16, 19]) which are mainly based on branch and bound procedures. In [3] a pseudo-approximation was presented. Earlier work include [18].

However when dealing with planar graphs with a single source and sink it was shown that by using planar duality, pseudo-polynomial algorithms for the network interdiction problem can be constructed when only arc removal is allowed [15]. Two of the major drawbacks of these algorithms (apart from the fact that they can only be applied on planar graphs) are the restrictions that only arc removals are allowed and that the network must have exactly one source and one sink. Vertex removal can easily be formulated as edge removal by a standard technique of doubling vertices and multiple sources and sink are generally handled by the

¹The term *network inhibition* is also used.

introduction of a supersource and supersink [1, 6]. However, these transformations destroy planarity and make it impossible to profit from the currently known specialized interdiction algorithms for planar graphs.

In this work, we are interested in the development of pseudo-polynomial algorithms for planar graphs which overcome some of the restrictions of previous methods. Inspired by algorithms presented in [10, 14], we propose a planarity-preserving transformation that allows to incorporate vertex removals and vertex capacities in pseudo-polynomial interdiction algorithms for planar graphs. Additionally, a pseudo-polynomial algorithm is introduced for the problem of determining the minimal interdiction budget needed to make it impossible to satisfy the demand of all sink nodes, on planar networks with multiple sources and sinks satisfying that the sum of the supplies at the source nodes equals the sum of the demands at the sink nodes.

This problem is closely related to the problem of determining if a flow network is $n - k$ secure, i.e. any removal of k of its components does not infect the value of the maximum flow, and can be seen as a special case of network interdiction.

It is not known if network interdiction on planar networks with multiple sources and sinks is a strongly NP-complete problem. We show that the k -densest subgraph problem on planar graphs, a problem for which neither a polynomial algorithm is known nor is it known if it is NP-complete, can be reduced to a planar network interdiction problem with polynomially bounded numbers as input.

The paper is organized as follows. We begin by giving some definitions and notations in Section 2. In Section 3, different known complexity results on network interdiction are stated and we show how the k -densest subgraph problem on planar graphs can be reduced to a planar network interdiction problem with small input numbers. Section 4 presents a simple extension of currently known algorithms for network interdiction problems on undirected networks where only arc removals are allowed to the case of directed networks. We present in Section 5 a pseudo-polynomial algorithm for network interdiction on planar $s-t$ flow networks allowing vertex interdiction. In Section 6 we present a pseudo-polynomial algorithm for a special network flow interdiction problem with multiple sources and sinks.

2 Definitions and Notations

Let (V, E) be a directed graph where V is the set of vertices, E is the set of arcs and for every arc $e \in E$, $u(e) \in \{0, 1, 2, \dots\}$ denotes its capacity. Two special nodes $s, t \in V, s \neq t$ designate the source node respectively the sink node (the generalization to multiple sources and sinks is straightforward). We call the network $G = (V, E, u, s, t)$ a flow network. For $V', V'' \subset V$ we denote by (V', V'') the set of all arcs from V' to V'' . Furthermore, for $V' \subset V$ we denote by $\omega^+(V')$ respectively $\omega^-(V')$ the set of all arcs exiting V' respectively entering V' , i.e., $\omega^+(V') = (V', V \setminus V')$ and $\omega^-(V') = (V \setminus V', V')$ ². For any subset V' of V we denote by $[V', V \setminus V']$ the cut defined by V' . The value of the cut $[V', V \setminus V']$ is $\sum_{e \in \omega^+(V')} u(e)$. In the more general setting when every arc $e \in E$ has an additional lower bound $l(e)$ on the capacity, the value of the cut $[V', V \setminus V']$ is defined by $\nu([V', V \setminus V']) = \sum_{e \in \omega^+(V')} u(e) - \sum_{e \in \omega^-(V')} l(e)$ ³. A cut $[V', V \setminus V']$ in G is called *elementary* if the subgraph of G induced by V' is connected. For two distinct vertices $s, t \in V$, a cut $[V', V \setminus V']$ is called an $s-t$ cut if $s \in V', t \notin V'$.

A function $f : E \rightarrow \mathbb{R}$ is called a flow in G (or simply flow if there is no danger of ambiguity) if it satisfies the following constraints:

- i) $0 \leq f(e) \leq u(e) \quad \forall e \in E$
- ii) $\sum_{e \in \omega^+(v)} f(e) - \sum_{e \in \omega^-(v)} f(e) = 0 \quad \forall v \in V \setminus \{s, t\}$
- iii) $\sum_{e \in \omega^+(s)} f(e) - \sum_{e \in \omega^-(s)} f(e) \geq 0$.

For an $s-t$ flow f we define its value $\nu(f)$ by $\sum_{e \in \omega^+(s)} f(e) - \sum_{e \in \omega^-(s)} f(e)$. A maximum $s-t$ flow is an $s-t$ flow f with maximum value. The value of a maximum $s-t$ flow in a flow network G is denoted by $\nu^{\max}(G)$.

In the context of network interdiction, for every arc and node of the network $p \in V \cup E$ an interdiction cost $c(p) \in \{1, 2, 3, \dots\} \cup \{\infty\}$ is associated (with $c(s) = c(t) = \infty$). The network $G = (V, E, u, s, t, c)$ is called an interdiction network. An interdiction network has unit interdiction costs if $c(p) \in \{1, \infty\} \forall p \in V \cup E$.

²We also use the notation ω_G^+ and ω_G^- to specify that we are working on the graph G .

³Sometimes the notation ν^G is used to specify the network.

The network interdiction problem asks to find a set $R \subset V \cup E$ respecting a given budget constraint $c(R) := \sum_{r \in R} c(r) \leq B$ (with $B \in \{0, 1, 2, \dots\}$), and among all these sets minimizing the value of a maximum s - t flow on the graph $G \setminus R$, which is the subgraph of G obtained by removing the arcs and vertices contained in R (when removing a vertex, all arcs adjacent to this vertex are removed too). The value of this minimum maximum s - t flow corresponding to budget B is denoted by $\nu_B^{\max}(G)$ (we therefore have $\nu^{\max}(G) = \nu_0^{\max}(G)$). In this context a set $R \subset V \cup E$ satisfying the budget constraint will be called an interdiction set. An optimal interdiction set R minimizes the maximum s - t flow with respect to the given budget. Furthermore an optimal interdiction set R is called minimum if its interdiction cost $c(R)$ is minimum among all optimal interdiction sets and it is called minimal when removing any arc from the interdiction set results in a non-optimal interdiction set.

We define the network flow security problem to be the problem of finding the minimal budget necessary to decrease the maximum flow by at least one unit, i.e., $\min\{B \in \{0, 1, 2, \dots\} \mid \nu_B^{\max}(G) < \nu^{\max}(G)\}$.

The above definitions and problems can easily be extended to interdiction networks with multiple sources and sinks with fixed supply/demand. In this case an interdiction network is given by $G = (V, E, u, S, T, c, d)$ where $S, T \subset V$ with $S \cap T = \emptyset$ are the set of sources respectively sinks and the function $d : V \rightarrow \mathbb{Z}$ is the demand/supply function and satisfies $d(s) < 0 \forall s \in S$, $d(t) > 0 \forall t \in T$ and $d(v) = 0 \forall v \in V \setminus (S \cup T)$.

To simplify notations a circuit C in G will be represented by the set of arcs it contains. For further graph-theoretical terms used in this paper and not further specified in this section we refer to [11].

3 Complexity

3.1 Previous results

We associate the following natural decision problems to the network interdiction problem respectively the network flow security problem.

Problem 1 (Decision version of network interdiction problem). *Given an interdiction network G , some interdiction budget $B \in \{0, 1, 2, \dots\}$ and a value $K \in \{0, 1, 2, \dots\}$, decide whether $\nu_B^{\max}(G) \leq K$?*

Problem 2 (Decision version of network flow security). *Given an interdiction network G and an interdiction budget $B \in \{0, 1, 2, \dots\}$, decide whether $\nu_B^{\max}(G) < \nu^{\max}(G)$?*

It is easy to observe that Problem 2 is a special case of Problem 1 by choosing $K = \nu^{\max}(G)$. Otherwise, when working on a class of interdiction networks with a single source or sink, Problem 1 can be reduced to Problem 2 by the following simple construction. Suppose we have a single source s (the case of a single sink is analogue). We introduce a new vertex s' which replaces s as source and add an unremovable arc from s' to s with capacity equal to K . Problem 2 on the modified interdiction network is then equivalent to Problem 1 on the initial interdiction network.

The following theorem was shown in [19] by reducing a maximum clique problem to Problem 1.

Theorem 1 ([19]). *Problem 1 is strongly NP-complete even when the underlying interdiction network is restricted to unit interdiction costs.*

Furthermore there is a trivial reduction from the binary knapsack problem (see [7] for more information on the binary knapsack problem) to an interdiction problem on a graph with only two vertices [19] implying the following theorem.

Theorem 2 ([19]). *Problem 1 is weakly NP-complete on planar graphs even when restricted to a single source and sink.*

A pseudo-polynomial algorithm for network interdiction problems on planar graphs with a single source and sink [15] show that this class of problems is not strongly NP-complete.

By the reducibility of Problem 1 to Problem 2, Theorem 1 and Theorem 2 apply also for Problem 2 when working on interdiction networks with a single source or a single sink.

It is not known whether the class of interdiction problems on planar graphs with multiple sources and sinks is strongly NP-complete. Furthermore, the direct reduction from Problem 1 to Problem 2 is not possible anymore on this class of networks. We will introduce in Section 6 a pseudo-polynomial algorithm

for Problem 2 on the class of planar interdiction networks with multiple sources and sinks and such that the sum of the demands is equal to the sum of the supplies and equal to the maximum flow in the initial network. This algorithm does not seem to generalize in a simple way to Problem 1.

3.2 Relation between planar network interdiction and the k -densest subgraph problem in planar graphs

We will show that finding dense subgraphs of a given size on planar graphs can easily be modelled as a planar network interdiction problem. The problem of finding a densest subgraphs of size k is often called the k -densest subgraph problem or the k -clustering problem and is formally defined as follows.

Problem 3 (k -densest subgraph problem). *Given an undirected graph $G = (V, E)$, find a subgraph over k vertices with a maximum number of edges.*

Whereas Problem 3 is known to be NP-complete on a wide variety of graph classes [4], its complexity for the class of planar graphs is still open. A slight modification of Problem 3 obtained by imposing that the subgraph must be connected was shown to be NP-complete on planar graphs [9].

Theorem 3. *The k -densest subgraph problem on planar graphs can be reduced in polynomial time to a network interdiction problem on planar graphs.*

Proof. Let $G = (V, E)$ be a planar undirected graph. Consider the following planar interdiction network $G' = (V', E', u, S, T, c, d)$. The underlying graph (V', E') is obtained from G by subdividing all edges, i.e., on every edge $e \in E$, a new node v_e is added. We thus obtain a bipartite planar graph where each edge has one endpoint in V and the other one in the set of newly added vertices V_E ($V' = V \cup V_E$). By directing all edges from V to V_E , we get E' (c.f. Figure 1). The sets containing the sources and sinks are defined as follows $S = V$, $T = V_E$, all arcs have unit capacity $u(e) = 1 \forall e \in E'$, sources have infinite supply $d(s) = \infty \forall s \in S$ and every sink has unit demand $d(t) = 1 \forall t \in T$. Furthermore all arcs and all vertices of V_E are unremovable (they have an interdiction cost of ∞) and the vertices in V have an interdiction cost equal to one.

For some fixed budget $B \in \{0, 1, 2, \dots\}$, an optimal interdiction set G' corresponds exactly to the vertices of a B -densest subgraph in G because of the following observation. For some interdiction set R , the decrease of flow by removing the components in R corresponds to the number of sinks for which both neighbors are in R . This corresponds to the number of edges in G that have both endpoints in R . \square

4 Planar duality and current pseudo-polynomial algorithms

Planarity is a very helpful property when dealing with interdiction problems as the problem seems to have a simpler form when restated on the planar dual of the original interdiction network. We first introduce the planar dual of an interdiction network, which can be seen as a generalization of the classical planar dual. In a second step we propose a pseudo-polynomial algorithm for planar network interdiction with a single source and a single sink and without vertex removals. This algorithm is a direct generalization of an algorithm introduced in [15], which was designed only for undirected networks⁴. The extensions we propose in the following sections will overcome some restrictions of this algorithm.

4.1 Planar duality for interdiction networks

The classical planar dual⁵ of a directed graph is constructed on the base of a planar embedding by placing a vertex in each face of the original graph and connecting two vertices by an arc if they correspond to faces in the original graph sharing an arc. This gives a natural one-to-one correspondence between arcs in the original graph and arcs in the dual graph (dual arcs) as well as faces in the original graph and vertices in the dual graph, and vice versa. By convention, the dual arcs are oriented such that they cross the corresponding original arcs from right to left. See [12] for more details.

⁴Furthermore the algorithm we present does not allow partial arc removals whereas the algorithm presented in [15] did allow it. This makes no real difference as the technique applies easily to both cases.

⁵It is also called *geometric dual* or simply *dual*.

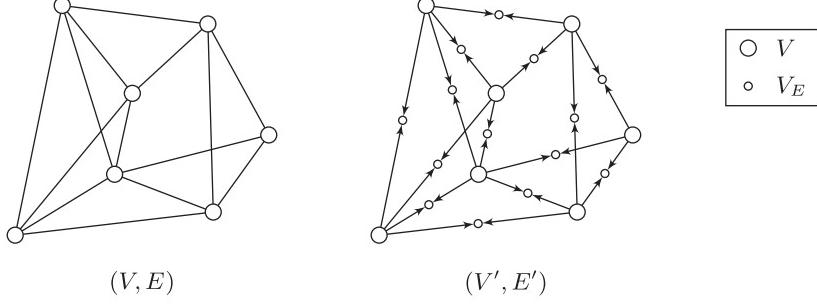


Figure 1: Topology of the auxiliary graph (V', E') used for proving Theorem 3.

For our purposes we extend the notion of planar duality on networks with lower and upper capacities on the arcs and interdiction costs on the arcs. Let $G = (V, E, l, u, c)$ be a directed planar network where for every arc $e \in E$, $l(e), u(e), c(e) \in \{0, 1, 2, \dots\}$ correspond to the lower capacity bound, upper capacity bound and interdiction cost of arc e (where $l(e) \leq u(e) \forall e \in E$). We define the dual $G^* = (V^*, E^*, \lambda^*, c^*)$ of the network G in the following way. The graph (V^*, E^*) is the planar dual in the classical sense of the graph (V, E) with the single difference that for every arc in the dual we added a reverse arc. For every arc $e \in E$ we denote by e^D the corresponding dual arc (as in the classical sense) and by e_R^D its reverse arc (cf. Figure 2). The function $\lambda^* : E^* \rightarrow \mathbb{Z}$ is an integral length function in the network G^* , defined by $\lambda^*(e^D) = u(e)$ and $\lambda^*(e_R^D) = -l(e) \forall e \in E$. The cost function c^* is defined by $c^*(e^D) = c(e), c^*(e_R^D) = 0 \forall e \in E$.

For every cut $[V', V \setminus V']$ in the original network we denote its corresponding dual arcs by $\mathcal{C}^*(V') = \{e^D \in E^* \mid e \in (V', V \setminus V')\} \cup \{e_R^D \in E^* \mid e \in (V \setminus V', V')\}$. Note that the set $\mathcal{C}^*(V')$ is a set of edge-disjoint, counterclockwise non-overlapping circuits in (V^*, E^*) , where non-overlapping is defined as follows. Let $\mathcal{C}_1^*, \mathcal{C}_2^*$ be two circuits in G^* and $V_1, V_2 \subset V$ be the vertices in V surrounded in counterclockwise sense by \mathcal{C}_1^* respectively \mathcal{C}_2^* . We say that $\mathcal{C}_1^*, \mathcal{C}_2^*$ do not overlap if $V_1 \cap V_2 = \emptyset$. The following proposition highlights the correspondence of minimal cuts in the network G and sets of non-overlapping, edge-disjoint circuits in its dual G^* .

Proposition 1. *The function that associates with every cut $[V', V \setminus V']$ its corresponding dual arcs $\mathcal{C}^*(V')$ is a one-to-one mapping between cuts in G and sets of non-overlapping, edge-disjoint, counterclockwise circuits in G^* . Furthermore, the value of a cut in G is equal to the sum of the lengths of the corresponding dual arcs in G^D , i.e., for any subset $V' \subset V$, we have*

$$\nu([V', V \setminus V']) = \sum_{e^* \in \mathcal{C}^*(V')} \lambda^*(e^*) .$$

Proof. The one-to-one property follows easily by observing that for any set \mathcal{C}_0^* of non-overlapping, edge-disjoint, counterclockwise circuits, the set V' of all vertices being surrounded in counterclockwise sense by one of the circuits in \mathcal{C}_0^* satisfies $\mathcal{C}^*(V') = \mathcal{C}_0^*$. The equality between the value of a cut in G and the sum of the lengths of the corresponding dual arcs follows directly from the definition of λ^* . \square

Proposition 1 implies that for every minimal cut in G , there exists a corresponding circuit in the dual G^* with length equal to the value of the cut. In particular, when working on a flow network $G = (V, E, l, u, s, t)$ with a single source s and a single sink t , every minimal $s-t$ cut in G corresponds to a counterclockwise circuit around s with t separated from s by the circuit and with length equal to the value of the cut. We call circuits that separate s and t in such a way $s-t$ separating counterclockwise circuits. In the following we see how this correspondence can be exploited for solving network interdiction problems on planar graphs with a single source and a single sink.

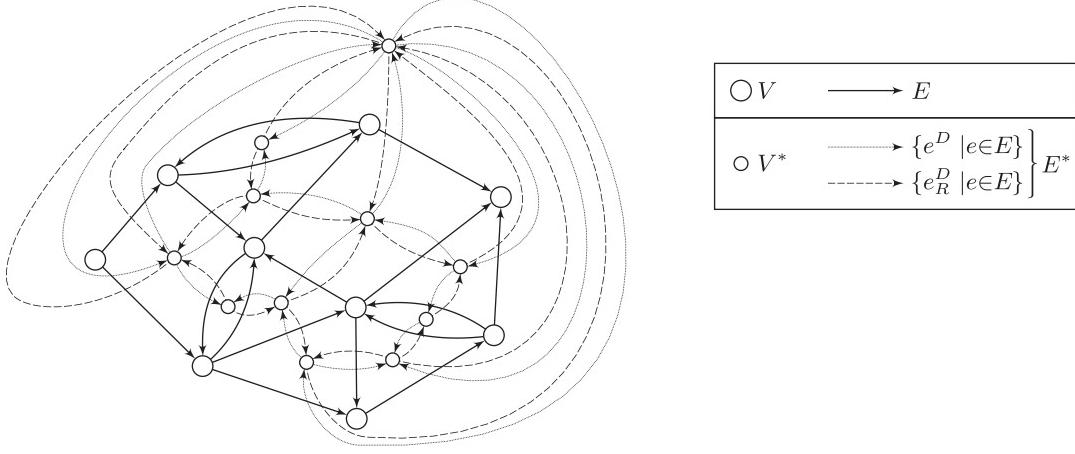


Figure 2: Example dual graph (V^*, E^*) drawn over the given original graph (V, E) .

4.2 A pseudo-polynomial algorithm for single source, single sink network interdiction on planar graphs without vertex removal

We now construct a pseudo-polynomial algorithm for solving the network interdiction problem on planar directed graphs with a single source s and a single sink t and without vertex removal, which is a direct generalization of an algorithm presented in [15] (that only worked for undirected graphs). This algorithm nicely illustrates the techniques currently used for creating pseudo-polynomial network interdiction algorithms on planar graphs. Given is an interdiction network $G = (V, E, u, s, t, c)$ with unremovable vertices, i.e., $c(r) = \infty \forall r \in V$, and interdiction budget B . For every interdiction set $R \subset E$ we fix a minimum s - t cut in $G \setminus R$ that we denote by $[V_R, V \setminus V_R]$. We therefore have $\nu^{\max}(G \setminus R) = \nu([V_R, V \setminus V_R]) - \sum_{e \in R \cap \omega^+(V_R)} c(e)$. Note that an optimal, minimal interdiction set R always satisfies $R \subset \omega^+(V_R)$, as otherwise the interdiction set $R' = R \cap \omega^+(V_R)$ would reduce the maximum flow by the same value as R and has lower interdiction cost. The reduced value of an s - t cut C (with respect to the budget B) is defined as the minimum value of C in $G \setminus R$ over all interdiction sets R and is denoted by $\nu_B(C)$. Note that the problem to find for some given s - t cut C an interdiction set R that minimizes the value of C in $G \setminus R$ is a binary knapsack problem.

The main idea of the algorithm is to find an optimal, minimum interdiction set R by finding a corresponding s - t cut with minimal reduced value. This is done by translating the problem into the dual. For any set of edges $U^* \subset E^*$, we define its reduced length (with respect to B) by $\lambda_B^*(U^*) = \min\{\sum_{e^* \in U^* \setminus X^*} \lambda^*(e^*) \mid X^* \subset U^*, \sum_{e^* \in X^*} c^*(e^*) \leq B\}$. Similarly for a walk W^* in G^* going along the arcs $(e_1^*, e_2^*, \dots, e_k^*)$, we define $\lambda_B^*(W^*) = \min\{\sum_{i \in \{1, 2, \dots, k\} \setminus I} \lambda^*(e_i^*) \mid I \subset \{1, 2, \dots, k\}, \sum_{i \in I} c^*(e_i^*) \leq B\}$. By the correspondence between s - t cuts in G and s - t separating counterclockwise circuits in G^* as highlighted in Section 4.1, we have that the problem of finding an s - t cut in G with minimal reduced value is equivalent to finding an s - t separating counterclockwise circuit with minimal reduced length in the dual. Such circuits can be described in the following way. Let P be any path in the graph G from vertex s to vertex t , we define $P^D = \{e^D \in E^* \mid e \in P\}$ and $P_R^D = \{e_R^D \in E^* \mid e \in P\}$. For any set of edges $U^* \subset E^*$ we define its parity with respect to P by $p_P(U^*) = |U^* \cap P^D| - |U^* \cap P_R^D|$. By a result of [14] we have that for every circuit C^* in G^* , two consecutive crossings on P alternate between left-right crossing and right-left ones. This implies that every circuit C^* in G satisfies $p_P(C^*) \in \{-1, 0, 1\}$. Another implication is that a circuit C^* in G^* has the properties to be counterclockwise s - t separating if and only if $p_P(C^*) = 1$.

We therefore have to solve the following problem.

Problem 4.

$$\operatorname{argmin}\{\lambda_B^*(C^*) \mid C^* \text{ circuit in } G^* \text{ with } p_P(C^*) = 1\}$$

Consider the following relaxation of Problem 4.

Problem 5.

$$\operatorname{argmin}\{\lambda_B^*(W^*) \mid W^* \text{ closed walk in } G^* \text{ with } p_P(W^*) = 1\}$$

A solution to Problem 4 can be easily obtained on the base of a solution W^* to Problem 5 by the following observation. W^* can be partitioned in a disjoint union of circuits $\mathcal{C}_1^*, \mathcal{C}_2^*, \dots, \mathcal{C}_k^*$. Furthermore, by modularity of the parity function p_P and the fact that $p_P(W^*) = 1$, we have $1 = p_P(W^*) = \sum_{i=1}^k p_P(\mathcal{C}_i^*)$. As the parity of each circuit is in $\{-1, 0, 1\}$ we have that there is some index $i \in \{1, 2, \dots, k\}$ with $p_P(\mathcal{C}_i^*) = 1$. From $\mathcal{C}_i^* \subset W^*$ follows that $\lambda_B^*(\mathcal{C}_i^*) \leq \lambda_B^*(W^*)$. By optimality of W^* for Problem 5 we thus have $\lambda_B^*(\mathcal{C}_i^*) = \lambda_B^*(W^*)$ and by the fact that Problem 5 is a relaxation of Problem 4 follows that \mathcal{C}_i^* is an optimal solution for Problem 4. More generally, the above reasoning shows that minimal solutions of Problem 5 correspond to solutions of Problem 4 and vice versa.

We finally show how Problem 5 can be solved by a dynamic programming approach that we realize as a sequence of shortest path problems on an auxiliary graph $G' = (V', E', \lambda')$ with positive arc lengths λ' which is defined as follows. Let P be a shortest path from s to t in G and we denote by $|P|$ the length of the path which is the number of arcs used in P . V' consists of $|V^*| \cdot (B + 1) \cdot (2|P| + 1)$ vertices that we denote in the following way $V' = \{v_{b,p}^* \mid v^* \in V^*, b \in \{0, 1, \dots, B\}, p \in \{-|P|, -|P| + 1, \dots, |P|\}$. The set of arcs E' is defined by $E' = E'_0 \cup E'_{1,=} \cup E'_{1,+} \cup E'_{1,-} \cup E'_{2,=} \cup E'_{2,+} \cup E'_{2,-}$ with

- $E'_0 = \{(v_{b,p}^*, v_{b-1,p}^*) \mid v_{b,p}^* \in V', b \in \{1, 2, \dots, B\}, p \in \{-|P|, -|P| + 1, \dots, |P|\}\}$
- $E'_{1,=} = \{(u_{b,p}^*, v_{b,p}^*) \mid (u^*, v^*) \in E^* \setminus (P^D \cup P_R^D), b \in \{0, 1, \dots, B\}, p \in \{-|P|, -|P| + 1, \dots, |P|\}\}$
- $E'_{1,+} = \{(u_{b,p}^*, v_{b,p+1}^*) \mid (u^*, v^*) \in P^D, b \in \{0, 1, \dots, B\}, p \in \{-|P|, -|P| + 1, \dots, |P| - 1\}\}$
- $E'_{1,-} = \{(u_{b,p}^*, v_{b,p-1}^*) \mid (u^*, v^*) \in P_R^D, b \in \{0, 1, \dots, B\}, p \in \{-|P| + 1, -|P| + 2, \dots, |P|\}\}$
- $E'_{2,=} = \{(u_{b_1,p}^*, v_{b_2,p}^*) \mid (u^*, v^*) \in E^* \setminus (P^D \cup P_R^D), b_1 \in \{c^*(u^*, v^*), c^*(u^*, v^*) + 1, \dots, B\}, b_2 = b_1 - c^*(u^*, v^*), p \in \{-|P|, -|P| + 1, \dots, |P|\}\}$
- $E'_{2,+} = \{(u_{b_1,p}^*, v_{b_2,p+1}^*) \mid (u^*, v^*) \in P^D, b_1 \in \{c^*(u^*, v^*), c^*(u^*, v^*) + 1, \dots, B\}, b_2 = b_1 - c^*(u^*, v^*), p \in \{-|P|, -|P| + 1, \dots, |P| - 1\}\}$
- $E'_{2,-} = \{(u_{b_1,p}^*, v_{b_2,p-1}^*) \mid (u^*, v^*) \in P_R^D, b_1 \in \{c^*(u^*, v^*), c^*(u^*, v^*) + 1, \dots, B\}, b_2 = b_1 - c^*(u^*, v^*), p \in \{-|P| + 1, -|P| + 2, \dots, |P|\}\}$.

We define a function $\eta : V' \cup (E' \setminus E'_0) \rightarrow V \cup E$ that maps elements of G' to corresponding elements in G^* in the following way.

$$\begin{aligned} \eta(v_{b,p}^*) &= v^* & \forall v_{b,p}^* \in V' \\ \eta(v_{b_1,p_1}^*, u_{b_2,p_2}^*) &= (v^*, u^*) & \forall (v_{b_1,p_1}^*, u_{b_2,p_2}^*) \in E' \setminus E'_0 \end{aligned}$$

The length $\lambda' : E' \rightarrow \{0, 1, \dots\}$ is defined as follows.

$$\lambda'(e') = \begin{cases} 0 & \text{if } e' \in E'_0 \cup E'_{2,=} \cup E'_{2,+} \cup E'_{2,-} \\ \lambda^*(\eta(e')) & \text{if } e' \in E'_{1,=} \cup E'_{1,+} \cup E'_{1,-} \end{cases}$$

We now define a correspondence between walks in G' and walks in G^* by extending the function η in the following way. Let $u_{b_1,p_1}^*, v_{b_2,p_2}^* \in V'$, W' be a walk in G' from u_{b_1,p_1}^* to v_{b_2,p_2}^* and $(e'_1, e'_2, \dots, e'_k)$ the suite of edges corresponding to the walk W' . Furthermore, let $I = \{i \in \{1, 2, \dots, k\} \mid e'_i \in E' \setminus E'_0\}$. The walk $\eta(W')$ in G' is defined to go along the edges $(\eta(e'_i))_{i \in I}$. It is easy to verify that $\eta(W')$ is effectively a walk (from u^* to v^*) in G^* and satisfies $p_P(\eta(W')) = p_2 - p_1$. We associate to the walk W' a set $R(W') \subset E$ defined by $R(W') = \{e \in E \mid \exists i \in I \text{ with } e'_i \in E'_{2,=} \cup E'_{2,+} \cup E'_{2,-} \text{ and } e^D = \eta(e'_i)\}$. By construction of G' we have $c(R(W')) \leq b_1 - b_2$ ⁶. Therefore, for any path W' in G' , the set $R(W')$ is an interdiction set. Furthermore, we have $\lambda_{b_1-b_2}^*(\eta(W')) \leq \lambda'(W')$ because by setting the length of the arcs $R(W')^D$ in G^* to zero, the length of $\eta(W')$ is smaller or equal than $\lambda'(W')$ ⁷.

In particular, if the walk W' goes from $v_{B,0}^*$ to $v_{0,1}^*$ for some $v^* \in V^*$, we then have that $\eta(W')$ is a closed walk in G^* with $p_P(\eta(W')) = 1$ and satisfying $\lambda_B^*(\eta(W')) \leq \lambda'(W')$. Conversely, for every circuit C^* in G^* satisfying $p_P(C^*) = 1$, we can find a corresponding path P' in G' with $\lambda'(P') = \lambda_B^*(C^*)$ in the following way. Let $(e_1^*, e_2^*, \dots, e_k^*)$ be a suite of arcs in E^* corresponding to the circuit C^* and $v^* \in V^*$ be

⁶When $\eta(W')$ corresponds to a path or a circuit, this relation is always satisfied with equality.

⁷Here too, in case that $\eta(W')$ corresponds to a path or a circuit, we have $\lambda_{b_1-b_2}^*(\eta(W')) = \lambda'(W')$.

the vertex corresponding to the tail of e_1^* . Let $I \subset \{1, 2, \dots, k\}$ be a set satisfying $\sum_{i \in I} c^*(e_i^*) \leq B$ and $\lambda_B^*(C^*) = \sum_{i \in I} \lambda^*(e_i^*)$.

The path P' will be defined by its corresponding suite of arcs $(e'_1, e'_2, \dots, e'_{k'})$ as follows. We set $k' = k + B - \sum_{i \in I} c^*(e_i^*)$. P' starts at the vertex $v_{B,0}^*$. The path P' will be defined by adding edges step by step. For $i \in \{1, 2, \dots, k\}$ let $e_i^* = (u^*, v^*)$ and $u_{b,p}^*$ be the endpoint of the currently constructed path. If $i \in I$, we set $e'_i = (u_{b,p}^*, v_{b-c^*(e_i^*), p+p_P(e_i^*)}^*)$ and otherwise $e'_i = (u_{b,p}^*, v_{b,p+p_P(e_i^*)}^*)$. For $i \in \{k+1, k+2, \dots, k'\}$, we set $e'_i = (v_{k'-i+1,1}^*, v_{k'-i,1}^*)$. It is easy to verify that P' is effectively a path in G' with $\eta(P') = C^*$ satisfying $\lambda'(P') = \lambda_B^*(C^*)$.

Let $v^* \in V^*$ be some fixed vertex and P' be a shortest path from $v_{B,0}^*$ to $v_{0,1}^*$. The discussion above shows that $\eta(P')$ is a solution to the following problem.

Problem 6.

$$\operatorname{argmin}\{\lambda_B^*(W^*) \mid W^* \text{ closed walk in } G^* \text{ containing vertex } v^* \text{ and satisfying } p_P(W^*) = 1\}$$

Solving Problem 6 for all vertices $v^* \in V^*$ will solve Problem 5. However, as a walk solving Problem 5 will pass at least once by an arc e' with $\eta(e') \in P^D$ it suffices to solve Problem 6 for all vertices in G^* having at least one outgoing edge contained in P^D . Therefore, it suffices to solve $|P|$ instances of Problem 6 to get a solution to Problem 5. Notice that solving the network interdiction problem with the above algorithm for some budget B , solves also the interdiction problems for all budgets $B' < B$. This solution can be obtained in an analogue manner as for the case with budget B by looking in G' at walks from $v_{B,0}^*$ to $v_{B-B',1}^*$.

Applying Dijkstra's algorithm for shortest paths to solve the subproblems of Problem 6, the algorithm introduced above to solve Problem 5 gets an overall complexity of $\mathcal{O}(B|P|^2 n \log(nB))$. Because of the special structure of the network G' it is easy slightly improve the running time by handling the vertices in Dijkstra's algorithm level by level with respect to the remaining budget, i.e., we handle all vertices corresponding to a budget B first (always taking the one with the smallest label as usual), then all with budget $B-1$ and so on. When looking for the vertex with the smallest label, we only have $O(n)$ candidates instead of $O(nB)$. This allows to reduce the running time to $\mathcal{O}(B|P|^2 n \log(n))$. By reversing the roles of budget and length in the proposed algorithm, one can replace B in the above running times by $\nu_B^{\max}(G)$.

Notice that the proposed method can be slightly simplified (without influencing the above running time bounds) allowing for the parity only to take values in $\{-\lceil \frac{|P|}{2} \rceil, -\lceil \frac{|P|}{2} \rceil + 1, \dots, \lceil \frac{|P|}{2} \rceil\}$ instead of $\{-|P|, -|P| + 1, \dots, |P|\}$ as every circuit in G^* with parity equal to one contains no subpath with a parity not contained in this range because it would not be possible to come back to parity one without going two times through a same vertex.

The formulation of Problem 6 as a shortest path problem on G' is essentially a dynamic programming realization of a multi-objective shortest path problem with the three objectives budget, length and parity on a graph \bar{G} defined as follows. The graph $\bar{G} = (V^*, \bar{E})$ is obtained from the graph $G^* = (V^*, E^*)$ by doubling every arc. For every arc $e^D \in E^*$, we denote by \bar{e}_1, \bar{e}_2 the two corresponding parallel arcs in \bar{E} . To every arc in \bar{E} we associate a parity value, a length and a budget value. The parity value of \bar{e}_1 and \bar{e}_2 is set to $p_P(e)$. The length of \bar{e}_1 is equal to $u(e)$ and the length of \bar{e}_2 is set to zero. Finally, the budget value is set to zero for \bar{e}_1 and to $c(e)$ for \bar{e}_2 . Therefore, the arcs in \bar{E} indexed by one correspond to non-removed arcs and the ones indexed by two correspond to removed arcs. For some fixed vertex $v^* \in V^*$, a closed walk in \bar{G} containing v , having parity equal to one, a budget value bounded by B and with minimal length among all those closed walks corresponds exactly to a solution to Problem 6.

5 Incorporating vertex interdiction and vertex capacities

Let $G = (V, E, u, s, t, c)$ be an interdiction network. In this section we show how vertex interdiction and vertex capacities can be incorporated in the algorithm of the previous section. We begin by introducing the possibility of vertex interdiction and observe afterwards how vertex capacities can be added to the model. The role of the dual network G^* will be replaced by a modified dual \tilde{G}^* which allows to model vertex interdiction basically as arc interdiction. This technique was used in [10] for modelling vertex capacities in planar flow problems. As we will see in this section, this model can also be used for modelling vertex interdiction. The modified dual we present in this section and extend in the next one is slightly different from the one presented in [10], using some less artificially added vertices.

The modified dual network $\widetilde{G}^* = (\widetilde{V}^*, \widetilde{E}^*, \widetilde{\lambda}^*, \widetilde{c}^*)$ is an extended version of $G^* = (V^*, E^*, \lambda^*, c^*)$ with additional vertices and arcs defined as follows. For every vertex $v \in V$ we denote by $f^*(v)$ the face of G^* corresponding to v . Similarly, for every $v^* \in V^*$ we denote by $f(v^*)$ the face of G corresponding to v^* . The network \widetilde{G}^* consists of the vertices V^* and contains an additional vertex for every face of G^* . Because of the one-to-one relation between faces in G^* and vertices in G we set $\widetilde{V}^* = V^* \cup V$. The arc set \widetilde{E}^* is defined by $\widetilde{E}^* = E^* \cup \widetilde{E}^V$ where \widetilde{E}^V contains the two arcs (v, v^*) and (v^*, v) for every pair of $v \in V$ and $v^* \in V^*$, where $f(v^*)$ is a face adjacent to v (c.f. Figure 3). The length function $\widetilde{\lambda}^*$ is an extension of λ^* on the arcs \widetilde{E}^* giving a value of ∞ to all arcs in \widetilde{E}^V (or simply a high finite value assuring that the arc is never used in following applications of shortest path algorithms). Furthermore, the modified interdiction cost function \widetilde{c}^* is an extension of c^* on the arcs in \widetilde{E}^* defined as follows.

$$\begin{aligned}\widetilde{c}^*(e^*) &= c^*(e^*) \quad \forall e^* \in E^* \\ \widetilde{c}^*(v, v^*) &= 0 \quad \forall (v, v^*) \in \widetilde{E}^V \cap V \times V^* \\ \widetilde{c}^*(v^*, v) &= c(v) \quad \forall (v^*, v) \in \widetilde{E}^V \cap V^* \times V\end{aligned}$$

For some subset of arcs $\widetilde{U}^* \subset \widetilde{E}^*$ we define their reduced cost in \widetilde{G}^* with respect to the budget B by $\widetilde{\lambda}_B^*(\widetilde{U}^*) := \min\{\sum_{\widetilde{e}^* \in \widetilde{U}^* \setminus X^*} \widetilde{\lambda}^*(\widetilde{e}^*) \mid X^* \subset \widetilde{U}^*, \sum_{\widetilde{e}^* \in X^*} \widetilde{c}^*(\widetilde{e}^*) \leq B\}$. The main idea of this model lies in the following correspondence between interdiction sets in G and s - t separating counterclockwise circuits in \widetilde{G}^* .

Theorem 4.

- i) For every interdiction set $R \subset V \cup E$ with respect to the budget B , there is an s - t separating counterclockwise circuit $\widetilde{\mathcal{C}}^*$ in \widetilde{G}^* satisfying $\widetilde{\lambda}_B^*(\widetilde{\mathcal{C}}^*) \leq \nu^{\max}(G \setminus R)$.
- ii) For every s - t separating counterclockwise circuit in \widetilde{G}^* and every budget B , there is an interdiction set $R \subset V \cup E$ with respect to B satisfying $\nu^{\max}(G \setminus R) \leq \widetilde{\lambda}_B^*(\widetilde{\mathcal{C}}^*)$.

Proof. Let $R \subset V \cup E$ be an interdiction set with respect to some fixed budget B and $U \subset E$ be a set of arcs corresponding to a minimum s - t cut in $G \setminus R$. As there is no path from s to t in the graph $G \setminus (R \cup U)$, it is easy to see (by means of the Jordan curve theorem) that there is an s - t separating counterclockwise circuit $\widetilde{\mathcal{C}}^*$ in \widetilde{G}^* consisting only of arcs which are either adjacent to vertices in R or are dual arcs of arcs contained in R or U . We therefore have as desired $\widetilde{\lambda}_B^*(\widetilde{\mathcal{C}}^*) \leq \nu^{\max}(G \setminus R)$.

Conversely let $\widetilde{\mathcal{C}}^*$ be an s - t separating counterclockwise circuit in \widetilde{G}^* satisfying $\widetilde{\lambda}_B^*(\widetilde{\mathcal{C}}^*) < \infty$ (when the reduced dual length is equal to ∞ , the result follows trivially). Let $\widetilde{U}^* \subset \widetilde{E}^*$ be a solution of

$$\operatorname{argmin}_{\widetilde{X}^* \subset \widetilde{\mathcal{C}}^*} \left\{ \sum_{\widetilde{e}^* \in \widetilde{\mathcal{C}}^* \setminus \widetilde{X}^*} \widetilde{\lambda}^*(\widetilde{e}^*) \mid \widetilde{c}^*(\widetilde{X}^*) \leq B \right\}.$$

By definition of the reduced dual length we have $\widetilde{\lambda}_B^*(\widetilde{\mathcal{C}}^*) = \widetilde{\lambda}^*(\widetilde{\mathcal{C}}^* \setminus \widetilde{U}^*)$. Let $V_{\widetilde{\mathcal{C}}^*}$ be the subset of vertices in V through which the circuit $\widetilde{\mathcal{C}}^*$ passes and let $U^* = \widetilde{U}^* \cap E^*$. Because of $\widetilde{\lambda}_B^*(\widetilde{\mathcal{C}}^*) < \infty$ we have that all arcs of $\widetilde{\mathcal{C}}^*$ entering in one of the vertices in $V_{\widetilde{\mathcal{C}}^*}$ are contained in \widetilde{U}^* . The cost of \widetilde{U}^* can therefore be reformulated as follow.

$$\widetilde{c}^*(\widetilde{U}^*) = c^*(U^*) + c(V_{\widetilde{\mathcal{C}}^*})$$

Let $U = \{e \in E \mid e^D \in U^*\}$ and we define $R = V_{\widetilde{\mathcal{C}}^*} \cup U$. By the above equation and the definition of \widetilde{U}^* we have $c(R) = c^*(U^*) + c(V_{\widetilde{\mathcal{C}}^*}) = \widetilde{c}^*(\widetilde{U}^*) \leq B$ showing that R is an interdiction set with respect to the budget B . Let $E_{\widetilde{\mathcal{C}}^*}^* = \widetilde{\mathcal{C}}^* \cap E^*$ and $E_{\widetilde{\mathcal{C}}^*} = \{e \in E \mid e^D \in E_{\widetilde{\mathcal{C}}^*}^*\}$. The fact that $\widetilde{\mathcal{C}}^*$ is a counterclockwise, s - t separating circuit implies that there is no path from s to t in $G \setminus (V_{\widetilde{\mathcal{C}}^*} \cup E_{\widetilde{\mathcal{C}}^*})$. Therefore, removing the arcs $E_{\widetilde{\mathcal{C}}^*} \setminus U$ from $G \setminus R$ destroys all paths from s to t implying $\nu^{\max}(G \setminus R) \leq u(E_{\widetilde{\mathcal{C}}^*} \setminus U)$. The result is finally obtained by observing that $u(E_{\widetilde{\mathcal{C}}^*} \setminus U) = \widetilde{\lambda}^*(\widetilde{\mathcal{C}}^* \setminus \widetilde{U}^*) = \widetilde{\lambda}_B^*(\widetilde{\mathcal{C}}^*)$. \square

Theorem 4 implies that the problem of finding by how much the value of a maximum flow can be reduced through interdiction reduces to finding an s - t separating counterclockwise circuit with minimal reduced value

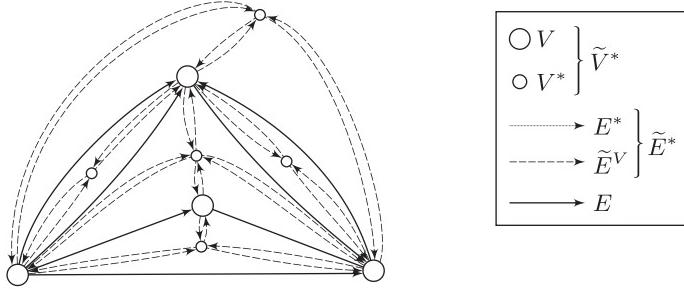


Figure 3: Topology of the auxiliary graph $(\tilde{V}^*, \tilde{E}^*)$ used for modeling vertex interdiction.

in \widetilde{G}^* . Furthermore, the proof of Theorem 4 shows how one can transform such a circuit to an optimal interdiction set.

For characterizing $s-t$ separating counterclockwise circuits, we introduce an adapted version of the parity function. As in the previous section, let P be a path in G from vertex s to vertex t . Because in the graph \widetilde{G}^* it is possible to cross P at a vertex, we have to take this possibility into account in the parity function. We therefore define a parity function \widetilde{p}_P which is an extension of p_P on the subsets of \widetilde{E}^* in the following way. For every vertex $v \in V \setminus \{s, t\}$ which lies on the path P and every arc $(v, v^*) \in \widetilde{E}^*$ which leaves P on the left side, we set $\widetilde{p}_P(v, v^*) = 1$. Similarly, for every vertex $v \in V \setminus \{s, t\}$ on P and every arc $(v^*, v) \in \widetilde{E}^*$ which enters P from the left side we set $\widetilde{p}_P(v^*, v) = -1$. For all other edges in \widetilde{E}^* we set $\widetilde{p}_P = 0$. Finally, for any set $\widetilde{U}^* \subset \widetilde{E}^*$ we define its parity by $\widetilde{p}_P(\widetilde{U}^*) = \sum_{e^* \in \widetilde{U}^*} \widetilde{p}_P(e^*)$. As in the previous section, we have, by this definition of \widetilde{p}_P , that a circuit $\widetilde{\mathcal{C}}^*$ in \widetilde{G}^* has the properties to be counterclockwise $s-t$ separating if and only if $\widetilde{p}_P(\widetilde{\mathcal{C}}^*) = 1$.

Applying the techniques of the previous section to the modified dual \widetilde{G}^* allows therefore to solve network interdiction problems on planar graphs with a single source and sink and with the possibility to interdict arcs and vertices. The asymptotic worst case complexity stays the same as in the previous section because the size of the graph \widetilde{G}^* is only at most a constant factor larger than G^* , and the same is true for the number of times Problem 5 has to be solved.

Upper capacities on the vertices can be introduced in the same way as shown in [10] by slightly modifying the network \widetilde{G}^* as follows. Suppose that some vertex $v \in V$ has an upper capacity $u(v) \in \{1, 2, \dots\}$. Let $V_v^* = \{v^* \in V^* \mid (v^*, v) \in \widetilde{E}^*\}$. For every $v^* \in V_v^*$, an additional arc from v^* to v is added with interdiction cost ∞ and capacity $u(v)$. For a justification of this construction in the case without interdiction (respectively by setting $B = 0$), see [10]. Theorem 4 remains true for the modified network modelling vertex capacities. This can be proven in an analogue way as for the case without vertex capacities. To simplify further results, we consider in the following only flow networks without vertex capacities. Vertex capacities can easily be added by the above construction.

The method presented in the next section to solve the network flow security problem on networks with multiple sources and sinks differs significantly from the approach presented in Section 4.2. Therefore the model presented in this section does not apply directly to this case. We will see in the next section how the model for vertex interdiction can be adapted (the adaption for vertex capacities will be analogue).

6 Pseudo-polynomial algorithm for network flow security

In this section we propose a pseudo-polynomial algorithm for the network flow security problem on planar graphs with multiple sources and sinks where the sum of the demands is equal to the sum of the supplies and equal to the maximum flow. The method is inspired by an algorithm introduced in [14] for testing if all demand can be satisfied in a planar flow problem with multiple sources and sinks. We will therefore begin with a short summary of this maximum flow algorithm for planar graphs in Subsection 6.1. Our algorithm for network flow security will be introduced in Subsection 6.2 restricted to the case without vertex removals.

This restriction will be lifted in Subsection 6.3 by adapting the modelling idea of Section 5.

6.1 Maximum flow algorithm of Miller and Naor

In this section we briefly present an algorithm introduced in [14] for testing if all demands can be satisfied in a flow problem with multiple sources and sinks such that the sum of all demands equals the sum of all supplies. Let $G = (V, E, l, u, S, T, d)$ be a planar flow network with lower and upper limits on the capacities designated by l and u , with source set $S \subset V$ and sink set $T \subset V \setminus S$ and with demand/supply function d satisfying $-d(S) = d(T)$. A flow in G is called *saturating* if it satisfies all demands. The problem we want to solve is the following.

Problem 7. Does there exist a saturating flow in G ?

In a first step, the problem is reduced to a circulation problem on a network \widehat{G} , which is the problem of finding a flow in a network with lower and upper capacities on the arcs but without sources and sinks. The network $\widehat{G} = (\widehat{V} = V, \widehat{E}, \widehat{l}, \widehat{u})$ is defined as follows. Let \mathcal{T}' be an undirected tree over the vertices in V that spans the sources and sinks and that can be added to G without destroying planarity. For every edge $\{v, u\} \in \mathcal{T}'$ we denote by $V_{\mathcal{T}'}(v, u) \subset V$ the set of vertices connected to vertex v in $\mathcal{T}' \setminus \{v, u\}$. We orient the edges in \mathcal{T}' to obtain \mathcal{T} in the following way. For $\{v, u\} \in \mathcal{T}'$ we orient the edge from v to u if $d(V_{\mathcal{T}'}(v, u)) \geq 0$, otherwise we orient the edge from u to v . The set \widehat{E} is defined to be $E \cup \mathcal{T}$. Furthermore, the lower and upper capacities \widehat{l} and \widehat{u} are extensions of l and u on the set \widehat{E} defined by $\widehat{l}(v, u) = \widehat{u}(v, u) = d(V_{\mathcal{T}'}(v, u)) \forall (v, u) \in \mathcal{T}$. As noted in [14] we have the following theorem.

Theorem 5. There exists a saturating flow in $G \Leftrightarrow$ there exists a circulation in \widehat{G} .

A circulation in a planar flow network can be computed by solving shortest path problem in its dual in the following way. Let $\widehat{G}^* = (\widehat{V}^*, \widehat{E}^*, \widehat{\lambda}^*)$ be the dual network of \widehat{G} as defined in Section 4, with the difference that we have no dual costs \widehat{c}^* as we deal with a standard flow network and not an interdiction network. Let $\widehat{r}^* \in \widehat{V}^*$ be an arbitrary vertex in \widehat{G}^* and for $\widehat{v}^* \in \widehat{V}^*$ let $\mu(\widehat{v}^*)$ be the distance of a shortest path from \widehat{r}^* to \widehat{v}^* (with respect to the length $\widehat{\lambda}^*$). In [14] the following theorem was proven.

Theorem 6. There exists a circulation in $\widehat{G} \Leftrightarrow \widehat{G}^*$ has no negative circuits.

In this case, a circulation can be obtained in the following way. Let $(\widehat{v}, \widehat{u}) \in \widehat{E}$ and $(\widehat{v}^*, \widehat{u}^*) = (\widehat{v}, \widehat{u})^D$ its corresponding dual arc. A circulation can finally be defined by assigning a flow equal to $\max\{0, \mu(\widehat{u}^*) - \mu(\widehat{v}^*)\}$ to each arc $(\widehat{v}, \widehat{u}) \in \widetilde{E}$.

6.2 Network security on planar graphs with multiple sources and sinks

Let $G = (V, E, l, u, S, T, d, c)$ be a planar interdiction network satisfying $-d(S) = d(T)$ and without vertex removal. Let $(\widehat{V}, \widehat{E}, \widehat{l}, \widehat{u})$ be the auxiliary network corresponding to (V, E, l, u, S, T, d) as defined in Section 6.1. We extend this auxiliary network to $\widehat{G} = (\widehat{V}, \widehat{E}, \widehat{l}, \widehat{u}, \widehat{c})$ where \widehat{c} is an extension of c on the set \widehat{E} , satisfying $\widehat{c}(\widehat{e}) = \infty \forall \widehat{e} \in \widehat{E} \setminus E$. Let $\widehat{G}^* = (\widehat{V}^*, \widehat{E}^*, \widehat{\lambda}^*, \widehat{c}^*)$ be its corresponding dual network as defined in Section 4. Using Theorem 6 we can formulate a problem on \widehat{G}^* , which is equivalent to Problem 2 for G , as follows.

Problem 8. Does there exist a circuit \widehat{C}^* in \widehat{G}^* with $\widehat{\lambda}_B^*(\widehat{C}^*) < \nu_B^{\max}(G)$?

The circuit with minimum reduced cost can be found by similar techniques as the ones presented in Section 4 with the differences that we do not have to take parity into account and that we have to use a shortest path algorithm, which can deal with negative arc lengths. A simple implementation would be first to determine shortest paths for all pairs of vertices in \widehat{G} . By using an algorithm presented in [5] this can be done in $\mathcal{O}(n^2 \log^3 n)$ time. To check if there is a circuit with negative reduced value in \widehat{G}^* going through some fixed vertex $\widehat{v}^* \in \widehat{V}^*$, a shortest path in an auxiliary network as described in Section 4 will be determined. This can be done by determining the shortest paths from some initial vertex to all others by proceeding level by level with respect to the remaining budget using at each level information of the preprocessing step. Over the B budget levels, this can be done in $\mathcal{O}(Bn^2)$ time. By simply performing this operation for all possible start vertices, we get a running time of $\mathcal{O}(Bn^3)$. By reversing the roles of budget and length, we can get an algorithm with running time $\mathcal{O}(\nu_B^{\max}(G)n^3)$. We expect that this running time can even be improved by exploiting more deeply the structure of the auxiliary graph.

6.3 Generalization to the case with vertex interdiction

Let $G = (V, E, l, u, S, T, d, c)$ be a planar interdiction network satisfying $-d(S) = d(T)$ and allowing arc and vertex removal (except for sources and sinks). As in the case without vertex removal, we begin by reformulating the problem as an interdiction problem for circulations. Let $\widehat{G} = (\widehat{V} = V, \widehat{E}, \widehat{l}, \widehat{u}, \widehat{c})$ be the auxiliary graph as defined in Section 6.2 and as before we denote by $\mathcal{T} = \widehat{E} \setminus E$ the added tree arcs. We now discuss how arc and vertex removal in G can be translated to \widehat{G} such that Theorem 5 remains valid for the resulting networks. As already exploited in Section 6.2, removing an arc of G corresponds to removing the same arc in \widehat{G} . However, vertex removal cannot be translated in such a direct way as the arcs in $\widehat{E} \setminus E$ are auxiliary arcs which should not be removed by a vertex removal. For any interdiction set $R \subset V \cup E$, we denote by $\widehat{G}(R)$ the graph obtained from \widehat{G} by removing all arcs contained in R and all arcs in E being adjacent to a vertex in R . We have the following relation.

Theorem 7. *For any interdiction set R of G we have the following equivalence.*

$$\text{There is a saturating flow in } G \setminus R \Leftrightarrow \text{there is a circulation in } \widehat{G}(R).$$

Proof. Let $G(R)$ be the network obtained from G by removing all arcs in R and all arcs adjacent to vertices in R . We trivially have that there is a saturating flow in $G \setminus R$ if and only if there is a saturating flow in $G(R)$. The network $\widehat{G}(R)$ can easily be obtained from $G(R)$ by applying the construction introduced in Section 6.1. Applying Theorem 5 proves finally the claim. \square

By a classical characterization for feasibility of circulation problems (see [1]) the following theorem follows.

Theorem 8. *For any interdiction set R in G we have:*

$$\text{There exists no circulation in } \widehat{G}(R) \Leftrightarrow \text{there exists a cut in } \widehat{G}(R) \text{ with value } < 0.$$

Furthermore, as the arcs contained in a cut can be partitioned in groups of arcs corresponding to elementary cuts, we have that there is a cut in $\widehat{G}(R)$ with value strictly less than zero exactly when there is an elementary cut in $\widehat{G}(R)$ with value strictly less than zero. In the following, we show how the problem of finding an interdiction set R and an elementary cut in $\widehat{G}(R)$ with negative value can be mapped onto a modified dual network of G . Let $\tilde{G}^* = (\tilde{V}^*, \tilde{E}^*, \tilde{\lambda}^*, \tilde{c}^*)$ be the modified dual network for the network \widehat{G} as introduced in Section 5⁸. Analogously as in Section 5, we use the notations $\tilde{V}^* = V \cup \widehat{V}^*$ and $\tilde{E}^* = \widehat{E}^* \cup \overline{E^V}$. We associate with every circuit $\mathcal{C} \in \tilde{G}$ a modified reduced length $\tilde{\gamma}_B^*(\mathcal{C})$ defined as follows. Let $V_{\tilde{C}^*}^I \subset V$ be all the vertices of V which are surrounded in counterclockwise sense by the circuit $\tilde{\mathcal{C}}^*$ in the graph \tilde{G}^* and let $V_{\tilde{C}^*} \subset V_{\tilde{C}^*}^I$ be the vertices of V on the circuit $\tilde{\mathcal{C}}^*$. $\mathcal{T}_{\tilde{C}^*}^+$ respectively $\mathcal{T}_{\tilde{C}^*}^-$ denote the set of arcs in \mathcal{T} going out respectively entering the set $V_{\tilde{C}^*}^I$ in \widehat{G} and being adjacent to one of the vertices in $V_{\tilde{C}^*}$, i.e.,

$$\begin{aligned}\mathcal{T}_{\tilde{C}^*}^+ &= \mathcal{T} \cap \omega_{\widehat{G}}^+(V_{\tilde{C}^*}^I) \cap \omega_{\widehat{G}}^+(V_{\tilde{C}^*}), \\ \mathcal{T}_{\tilde{C}^*}^- &= \mathcal{T} \cap \omega_{\widehat{G}}^-(V_{\tilde{C}^*}^I) \cap \omega_{\widehat{G}}^-(V_{\tilde{C}^*}).\end{aligned}$$

We finally define

$$\tilde{\gamma}_B^*(\tilde{\mathcal{C}}^*) = \tilde{\lambda}_B^*(\tilde{\mathcal{C}}^*) + \widehat{u}(\mathcal{T}_{\tilde{C}^*}^+) - \widehat{l}(\mathcal{T}_{\tilde{C}^*}^-). \quad (1)$$

The following theorem shows how the adapted reduced cost can be used to reformulate the interdiction problem.

Theorem 9. *Let B be a fixed budget. The following statements are equivalent.*

- i) *There exists an interdiction set R in \widehat{G} such that there exists an elementary cut $[V', V \setminus V']$ in $\widehat{G}(R)$ with $\nu^{\widehat{G}(R)}([V', V \setminus V']) < 0$.*
- ii) *There exists a circuit $\tilde{\mathcal{C}}^*$ in \tilde{G}^* with $\tilde{\gamma}_B^*(\tilde{\mathcal{C}}^*) < 0$.*

⁸Using the notation \tilde{G}^* instead of \tilde{G}^* would be more consistent. To simplify notations we chose the second form.

Proof. Let R be an interdiction set in \widehat{G} and $[V', V \setminus V']$ be an elementary cut in $\widehat{G}(R)$ with strictly negative value. We partition the set R into $R = V_R \cup \widehat{E}_R$ with $V_R = R \cap V$ and $\widehat{E}_R = R \cap \widehat{E}$. Let $U_+ = \mathcal{T} \cap \omega_{\widehat{G}}^+(V') \cap \omega_{\widehat{G}}^+(V_R)$ respectively $U_- = \mathcal{T} \cap \omega_{\widehat{G}}^-(V') \cap \omega_{\widehat{G}}^-(V_R)$ the subset of arcs in \mathcal{T} going out respectively entering into the set V' and being adjacent to vertices in V_R . Furthermore let $W = \omega_{\widehat{G} \setminus R}^+(V' \setminus R)$. The value of the cut $[V', V \setminus V']$ in $\widehat{G}(R)$ can be rewritten in the following way.

$$\nu^{\widehat{G}(R)}([V', V \setminus V']) = \widehat{u}(W) + \widehat{u}(U_+) - \widehat{l}(U_-)$$

Notice that in the network $\widehat{G} \setminus (R \cup W)$, there is no path from V' to vertices in $V \setminus V'$. This implies that we can find a counterclockwise circuit $\tilde{\mathcal{C}}^*$ in \tilde{G}^* consisting only of dual arcs of arcs in $\widehat{E}_R \cup W$ and of arcs being adjacent to vertices in V_R . Such a circuit furthermore satisfies $V_{\tilde{\mathcal{C}}^*}^I = V'$. This implies $U_+ = \mathcal{T}_{\tilde{\mathcal{C}}^*}^+$ and $U_- = \mathcal{T}_{\tilde{\mathcal{C}}^*}^-$. By a reasoning identical to the one in Section 5 we have $\lambda_B^*(\tilde{\mathcal{C}}^*) \leq \widehat{u}(W)$ implying finally $\tilde{\gamma}_B^*(\tilde{\mathcal{C}}^*) \leq \nu^{\widehat{G}(R)}([V', V \setminus V']) < 0$.

Conversely let $\tilde{\mathcal{C}}^*$ be a circuit in \tilde{G}^* with $\tilde{\gamma}_B^*(\tilde{\mathcal{C}}^*) < 0$. We will show how to find an interdiction set R and a cut $[V', V \setminus V']$ in $\widehat{G}(R)$ with $\nu^{\widehat{G}(R)}([V', V \setminus V']) < 0$. The existence of such a cut implies directly the existence of an elementary cut satisfying the claim. We set $V' = V_{\tilde{\mathcal{C}}^*}^I$ and $V_R = V_{\tilde{\mathcal{C}}^*}$. As before by defining $U_+ = \mathcal{T} \cap \omega_{\widehat{G}}^+(V') \cap \omega_{\widehat{G}}^+(V_R)$ and $U_- = \mathcal{T} \cap \omega_{\widehat{G}}^-(V') \cap \omega_{\widehat{G}}^-(V_R)$ we have $U_+ = \mathcal{T}_{\tilde{\mathcal{C}}^*}^+$ and $U_- = \mathcal{T}_{\tilde{\mathcal{C}}^*}^-$. Let $\widehat{E}_{\tilde{\mathcal{C}}^*}^* \subset \widehat{E}^*$ be the subset of all arcs in \widehat{E}^* which are contained in the circuit $\tilde{\mathcal{C}}^*$, this corresponds to the set of all arcs in $\tilde{\mathcal{C}}^*$ that are not adjacent to a vertex in V_R . Furthermore we define $\widehat{E}_{\tilde{\mathcal{C}}^*} = \{\tilde{e} \in \widehat{E} \mid \tilde{e}^D \in \widehat{E}_{\tilde{\mathcal{C}}^*}^*\}$. The minimal value of the cut $[V', V \setminus V']$ in $\widehat{G}(R)$ with respect to the interdiction set R can be reformulated as follows.

$$\begin{aligned} \min_{R \subset V \cup \widehat{E}: \widehat{c}(R) \leq B} \nu^{\widehat{G}(R)}([V', V \setminus V']) &\leq \min_{\widehat{E}_R \subset \widehat{E}: \widehat{c}(\widehat{E}_R) \leq B - \widehat{c}(V_R)} \nu^{\widehat{G}(V_R \cup \widehat{E}_R)}([V', V \setminus V']) \\ &= \nu_{B - \widehat{c}(V_R)}^{\widehat{G}}(E_{\tilde{\mathcal{C}}^*}) + \widehat{u}(U_+) - \widehat{l}(U_-) \\ &= \tilde{\lambda}_{B - \widehat{c}(V_R)}^*(\widehat{E}_{\tilde{\mathcal{C}}^*}^*) + \widehat{u}(U_+) - \widehat{l}(U_-) \end{aligned} \quad (2)$$

Note that we have furthermore $\tilde{\lambda}_{B - \widehat{c}(V_R)}^*(\widehat{E}_{\tilde{\mathcal{C}}^*}^*) = \tilde{\lambda}_B^*(\tilde{\mathcal{C}})$. This comes from the facts that all arcs $\tilde{e}^* \in \tilde{\mathcal{C}}^* \setminus \widehat{E}_{\tilde{\mathcal{C}}^*}^*$ satisfy $\tilde{\lambda}^*(\tilde{e}^*) = \infty$ and thus have to be interdicted in the calculation of $\tilde{\lambda}_B^*(\tilde{\mathcal{C}})$ and that $\tilde{c}^*(\tilde{\mathcal{C}}^* \setminus \widehat{E}_{\tilde{\mathcal{C}}^*}^*) = c(V_R)$ as already seen in Section 5. Further developing Equation 2 by using this relation, the definition of $\tilde{\gamma}_B^*$ and $U_+ = \mathcal{T}_{\tilde{\mathcal{C}}^*}^+$, $U_- = \mathcal{T}_{\tilde{\mathcal{C}}^*}^-$ we finally get

$$\min_{R \subset V \cup \widehat{E}: \widehat{c}(R) \leq B} \nu^{\widehat{G}(R)}([V', V \setminus V']) \leq \tilde{\gamma}_B^*(\tilde{\mathcal{C}}^*) < 0$$

proving the claim. \square

In the next step, we introduce a new length function $\tilde{\chi}^* : E^* \rightarrow \mathbb{Z} \cup \{\infty\}$, which is a slight adaption of $\tilde{\lambda}^*$ and satisfies $\tilde{\chi}_B^*(\tilde{\mathcal{C}}^*) = \tilde{\gamma}_B^*(\tilde{\mathcal{C}}^*)$ for every circuit $\tilde{\mathcal{C}}^*$ in \tilde{G}^* , where $\tilde{\chi}_B^*$ is the reduced length with respect to $\tilde{\chi}^*$ and the budget B . Such a length function allows us to solve the network security problem on G as in the previous sections by looking for a circuit in \tilde{G}^* with negative reduced length with respect to $\tilde{\chi}^*$.

Let $\tilde{\mathcal{C}}^*$ be a circuit in \tilde{G}^* . For every arc $\tilde{e}^* \in \tilde{E}^*$ we set $\tilde{\chi}^*(\tilde{e}^*) = \tilde{\lambda}^*(\tilde{e}^*)$. The value of $\tilde{\chi}^*$ on the arcs in \tilde{E}^V will be defined such that for each circuit $\tilde{\mathcal{C}}^*$ in \tilde{G}^* and each vertex $v \in V$ the following equality will be satisfied.

$$\tilde{\chi}^*(\tilde{\mathcal{C}}^* \cap \omega_{\tilde{\mathcal{C}}^*}(v)) = \tilde{\lambda}^*(\tilde{\mathcal{C}}^* \cap \omega_{\tilde{\mathcal{C}}^*}(v)) + \widehat{u}(\mathcal{T}_{\tilde{\mathcal{C}}^*}^+ \cap \omega_{\tilde{\mathcal{C}}^*}(v)) - \widehat{l}(\mathcal{T}_{\tilde{\mathcal{C}}^*}^- \cap \omega_{\tilde{\mathcal{C}}^*}(v)) \quad (3)$$

Summing the above equation over all vertices on the circuit $\tilde{\mathcal{C}}^*$ gives $\tilde{\chi}^*(\tilde{\mathcal{C}}^* \cap \tilde{E}^V) = \tilde{\lambda}^*(\tilde{\mathcal{C}}^* \cap \tilde{E}^V) + \widehat{u}(\mathcal{T}_{\tilde{\mathcal{C}}^*}^+) - \widehat{l}(\mathcal{T}_{\tilde{\mathcal{C}}^*}^-)$, which implies the desired property $\tilde{\chi}_B^*(\tilde{\mathcal{C}}^*) = \tilde{\gamma}_B^*(\tilde{\mathcal{C}}^*) \forall \tilde{\mathcal{C}}^* \text{ circuit in } \tilde{G}^*$. In the following, we define $\tilde{\chi}^*$ on the arcs \tilde{E}^V in such a way that Equation 3 is satisfied.

For all arcs $\tilde{e}^* \in \tilde{E}^V$ adjacent to a vertex in $S \cup T$ we set $\tilde{\chi}^*(\tilde{e}^*) = \tilde{\lambda}^*(\tilde{e}^*) = \infty$. This ensures that Equation 3 is satisfied for all $v \in S \cup T$. Let $v \in V \setminus (S \cup T)$, $\mathcal{T}^+(v)$ be all arcs in \mathcal{T} going out of v , \mathcal{T}_v^- be all arcs in \mathcal{T} entering v , $\mathcal{T}(v) = \mathcal{T}_v^+ \cup \mathcal{T}_v^-$ and let \tilde{E}_v^V be the set of all arcs in \tilde{E}^V adjacent to v . Furthermore,

let \tilde{e}_v^* be an arbitrary fixed arc in \tilde{E}_v^V . We denote by H the (geometric) graph $(\tilde{V}^*, \tilde{E}^* \cup \mathcal{T})$ obtained by adding the arcs \mathcal{T} to the planar graph $(\tilde{V}^*, \tilde{E}^*)$. For $\tilde{e}_1^*, \tilde{e}_2^* \in \tilde{E}_v^V$, we define $\mathcal{T}_v(\tilde{e}_1^*, \tilde{e}_2^*)$ to be the set of all arcs in \mathcal{T}_v that are traversed in H when going in counterclockwise sense from \tilde{e}_1^* around v to \tilde{e}_2^* . Furthermore, let $\mathcal{T}_v^+(\tilde{e}_1^*, \tilde{e}_2^*)$ respectively $\mathcal{T}_v^-(\tilde{e}_1^*, \tilde{e}_2^*)$ be the arcs in $\mathcal{T}_v(\tilde{e}_1^*, \tilde{e}_2^*)$, that are entering respectively going out of v . To simplify notations, we define $\alpha(\tilde{e}_1^*, \tilde{e}_2^*) = \hat{u}(\mathcal{T}_v^+(\tilde{e}_1^*, \tilde{e}_2^*)) - \hat{l}(\mathcal{T}_v^-(\tilde{e}_1^*, \tilde{e}_2^*))$. Finally we define for $\tilde{e}^* \in \tilde{E}_v^V$

$$\tilde{\chi}^*(\tilde{e}^*) = \begin{cases} \tilde{\lambda}^*(\tilde{e}^*) + \alpha(\tilde{e}^*, \tilde{e}_v^*) & \text{if } \tilde{e}^* \text{ enters } v \\ \tilde{\lambda}^*(\tilde{e}^*) - \alpha(\tilde{e}^*, \tilde{e}_v^*) & \text{if } \tilde{e}^* \text{ leaves } v . \end{cases}$$

Let $\tilde{\mathcal{C}}^*$ be a circuit in \tilde{G}^* and $v \in V$ a vertex though which $\tilde{\mathcal{C}}^*$ passes. We denote by \tilde{e}_{in}^* respectively \tilde{e}_{out}^* the incoming and outgoing arc of $\tilde{\mathcal{C}}^*$ with respect to the vertex v . Using the introduced notation, Equation 3 can be rewritten as

$$\tilde{\chi}^*(\tilde{e}_{\text{in}}^*) + \tilde{\chi}^*(\tilde{e}_{\text{out}}^*) = \tilde{\lambda}^*(\tilde{e}_{\text{in}}^*) + \tilde{\lambda}^*(\tilde{e}_{\text{out}}^*) + \alpha(\tilde{e}_{\text{in}}^*, \tilde{e}_{\text{out}}^*) . \quad (3')$$

Applying furthermore the definition of $\tilde{\chi}^*$ in Equation 3', we finally have to prove the following proposition.

Proposition 2.

$$\alpha(\tilde{e}_{\text{in}}^*, \tilde{e}_v^*) - \alpha(\tilde{e}_{\text{out}}^*, \tilde{e}_v^*) = \alpha(\tilde{e}_{\text{in}}^*, \tilde{e}_{\text{out}}^*)$$

Proof. We distinguish two cases describing two possible constellations of the arcs $\tilde{e}_v^*, \tilde{e}_{\text{in}}^*, \tilde{e}_{\text{out}}^*$ around the vertex v . When going in counterclockwise sense around v beginning at \tilde{e}_{in}^* , these arcs are either encountered in the order $(\tilde{e}_{\text{in}}^*, \tilde{e}_{\text{out}}^*, \tilde{e}_v^*)$ or $(\tilde{e}_{\text{in}}^*, \tilde{e}_v^*, \tilde{e}_{\text{out}}^*)$.

Case 1: $(\tilde{e}_{\text{in}}^*, \tilde{e}_{\text{out}}^*, \tilde{e}_v^*)$

The result follows immediately by the definition of α and the relations

$$\begin{aligned} \hat{u}(\mathcal{T}_v^+(\tilde{e}_{\text{in}}^*, \tilde{e}_{\text{out}}^*)) + \hat{u}(\mathcal{T}_v^-(\tilde{e}_{\text{out}}^*, \tilde{e}_v^*)) &= \hat{u}(\mathcal{T}_v^+(\tilde{e}_{\text{in}}^*, \tilde{e}_v^*)) \\ \hat{l}(\mathcal{T}_v^-(\tilde{e}_{\text{in}}^*, \tilde{e}_{\text{out}}^*)) + \hat{l}(\mathcal{T}_v^-(\tilde{e}_{\text{out}}^*, \tilde{e}_v^*)) &= \hat{l}(\mathcal{T}_v^-(\tilde{e}_{\text{in}}^*, \tilde{e}_v^*)) . \end{aligned}$$

Case 2: $(\tilde{e}_{\text{in}}^*, \tilde{e}_v^*, \tilde{e}_{\text{out}}^*)$

With the same reasoning as in the first case we get

$$\alpha(\tilde{e}_{\text{in}}^*, \tilde{e}_{\text{out}}^*) - \alpha(\tilde{e}_{\text{in}}^*, \tilde{e}_v^*) = \alpha(\tilde{e}_v^*, \tilde{e}_{\text{out}}^*) .$$

The proposition thus reduces to $\alpha(\tilde{e}_v^*, \tilde{e}_{\text{out}}^*) = -\alpha(\tilde{e}_{\text{out}}^*, \tilde{e}_v^*)$, which follows by observing that by construction of \mathcal{T} and the fact that $v \notin S \cup T$ we have

$$\hat{u}(\mathcal{T}_v^+) - \hat{l}(\mathcal{T}_v^-) = 0 .$$

□

The network security problem on G can thus be solved as in Section 6.2 by looking for a circuit $\tilde{\mathcal{C}}^*$ in \tilde{G}^* with negative reduced length with respect to $\tilde{\chi}^*$. If and only if such a circuit exists, it is possible to diminish the value of a maximum flow in the network G by at least one unit given the budget B . It follows from the proof of Theorem 9 that in this case, an interdiction strategy can be found in the same way as in presented in Section 5. As the network \tilde{G} used in this section is only at most a constant factor larger than the auxiliary network used in Section 6.2, the algorithm can be implemented with a running time of $\mathcal{O}(Bn^3)$ or, by reversing the roles of budget and length, with a running time of $\mathcal{O}(\nu_B^{\max}(G)n^3)$. Here, too, we expect that the running time can be improved by exploiting more deeply the structure of the network \tilde{G} in the dynamic programming steps.

7 Conclusions

We proposed a planarity-preserving transformation that allows to incorporate vertex removals and vertex capacities in pseudo-polynomial interdiction algorithms for planar graphs. Additionally, a pseudo-polynomial algorithm was introduced for the problem of determining the minimum interdiction budget needed to make

it impossible to satisfy the demand of all sink nodes. The algorithm works on planar networks with multiple sources and sinks where the sum of the supplies at the source nodes equals the sum of the demands at the sink nodes. However this algorithm does not seem to extend easily to general network interdiction problems with multiple sources and sinks. Thus, it is still not known whether network flow interdiction on planar graphs with multiple sources and sinks is solvable in pseudo-polynomial time. We showed that the k -densest subgraph problem on planar graphs can be polynomially reduced to a network flow interdiction problem on a planar graph with multiple sources and sinks. The algorithms presented in this paper can easily be adapted to the case when multiple resources are needed for removing arcs and nodes, still remaining pseudo-polynomial.

The main purpose of the algorithms presented in this paper, was to show that various interdiction problems on planar graphs can be solved in pseudo-polynomial time. We expect however, that it should be possible to speed up the proposed algorithms by using more elaborate techniques as for example nested dissection. Furthermore, it would be interesting to extend the presented work to nearly planar networks as for example networks with a bounded crossing number or genus.

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